Math 2010 May 16, 2016

§1. Quick Review of Vectors.

<u>Vector</u>: A vector is a directed line segment. In $\mathbb{R}^3$, we can write
(direction + length)

$$\vec{v} = (v_1, v_2, v_3), \quad v_1, v_2, v_3 \in \mathbb{R}$$

<u>Properties</u>:

- Scalar multiplication and addition.

<u>Example</u>: $(1, 2, \pi) + (\sqrt{2}, \frac{1}{2}, 5) = (1+\sqrt{2}, \frac{5}{2}, 5+\pi)$

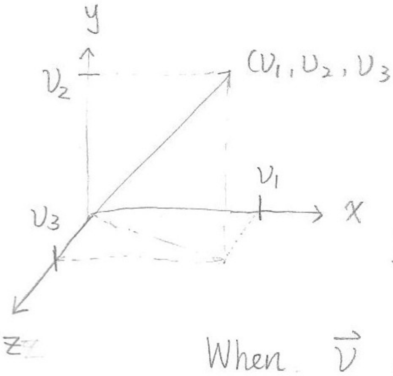
$4 \cdot (1, 2, \pi) = (4, 8, 4\pi)$

- <u>dot product</u>: Let $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$. Define

$$\vec{v} \cdot \vec{w} := v_1 w_1 + v_2 w_2 + v_3 w_3 .$$

<u>By Pythagorean theorem</u>: $\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 = (\text{length of } \vec{v})^2$

See Pic. 1.



Pic. 1

<u>Define</u>: $|\vec{v}| := \sqrt{\vec{v} \cdot \vec{v}}$

Then we have $|r\vec{v}| = |r| \cdot |\vec{v}|$ for $r \in \mathbb{R}$.

$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}|$

(Triangle inequality)

When $\vec{v}$ denotes the velocity (vector) $\Rightarrow$ $|\vec{v}|$ is the speed. (scalar quantity)

<u>Properties</u>: $\cdot$ $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cdot \cos\theta$

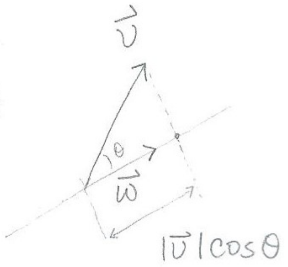
where θ is the angle between $\vec{v}$ and $\vec{w}$.

$$\left(\Rightarrow \cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} \right)$$

Properties (continued)

- Orthogonality : $\vec{v} \perp \vec{w} \Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \vec{v} \cdot \vec{w} = 0$
- Projection : $\vec{v} \cdot \vec{w} \cdot \dfrac{\vec{w}}{|\vec{w}|^2}$ is the projection of $\vec{v}$ on the line generated by $\vec{w}$

Pic. 2



Check : 1). direction = direction of $\vec{w}$

2) length $= |\vec{v}| \cdot |\vec{w}| \cos\theta \cdot \dfrac{|\vec{w}|}{|\vec{w}|^2}$

$= |\vec{v}| \cos\theta$ See Pic. 2

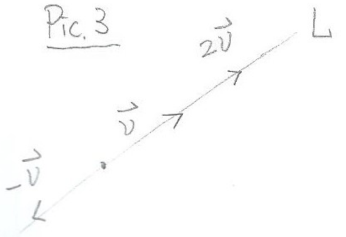
- Cauchy's inequality : $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| \cdot |\vec{w}|$
  (By using the fact that $|\cos\theta| \leq 1$)

Application of Dot product :

1. Equation of a line : We can generate a line by any vector $\vec{v}$ as follows :

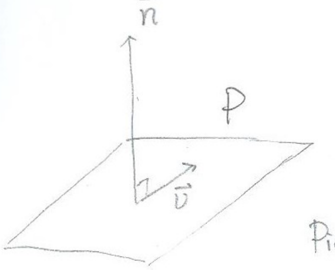
$$L := \{ r \cdot \vec{v} \mid r \in \mathbb{R} \}.$$ See Pic. 3

Pic. 3



2. Equation of a Plane : By choosing a normal vector $\vec{n}$, we can define a plane orthogonal to $\vec{n}$ as the follows :

$$P := \{ \vec{v} \mid \vec{v} \cdot \vec{n} = 0 \}.$$ See Pic. 4



Pic.4

Cross product: Let $\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$ $\bar{k} = (0,0,1)$.

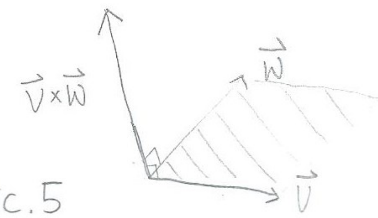
We define

$$\vec{v} \times \vec{w} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \bar{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \bar{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \bar{k}$$

(formally) (vector)

- Direction: (Right-Hand Rule) $\vec{v} \times \vec{w}$ will be perpendicular to the plane generated by $\vec{v}$ and $\vec{w}$ satisfying RHR.
- Length: Area of Parallelogram generated by $\vec{v}$ and $\vec{w}$.

See Pic. 5



Pic.5

Like

(Both Cross product and Facebook Like satisfy Right-Hand Rule).

Properties:

- Determinant: Let $\vec{u} = (u_1\ u_2\ u_3)$, $\vec{v}$, $\vec{w}$ be vectors.

$$\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \vec{u}\cdot(\vec{v}\times\vec{w})$$

This implies that $\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$

- By RHR, we have $\vec{v}\times\vec{w} = -\vec{w}\times\vec{v}$

Applications of Cross Product:

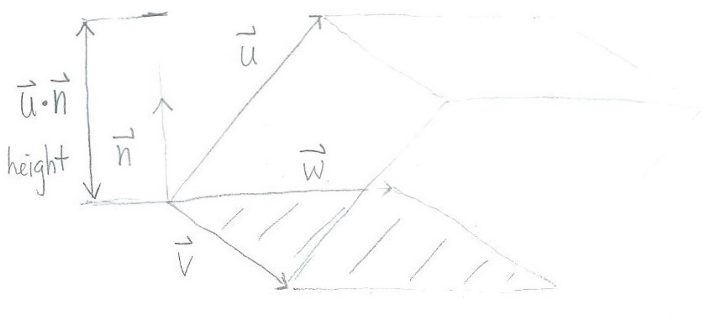
- Recall that, $\left|\det\begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix}\right|$ = volume of the parallelepiped generated by $\vec{u}, \vec{v}, \vec{w}$.

$$= |\vec{u}\cdot(\vec{v}\times\vec{w})|$$

Geometrically, we write $\vec{v}\times\vec{w} = |\vec{v}\times\vec{w}|\,\vec{n}$, where $\vec{n}$ is the unit normal vector.
$(|\vec{n}| = 1)$

So, $|\vec{u}\cdot(\vec{v}\times\vec{w})| = |(\vec{u}\cdot\vec{n})||(\vec{v}\times\vec{w})|$

$= \text{height}\cdot(\text{Area of parallelogram})$. See Pic. 6



Pic. 6.

- Let P_1, P_2, P_3 be 3 points in $\mathbb{R}^3$, then

(P is on the plane containing $P_1 P_2 P_3$)

$$\Leftrightarrow \left(\overrightarrow{PP_1}, \overrightarrow{PP_2}, \overrightarrow{PP_3} \text{ generate a parallelepiped of volume } 0 \right)$$

$$\Leftrightarrow \overrightarrow{PP_1} \cdot (\overrightarrow{PP_2} \times \overrightarrow{PP_3}) = 0$$

$$\Leftrightarrow \det \begin{pmatrix} \overrightarrow{PP_1} \\ \overrightarrow{PP_2} \\ \overrightarrow{PP_3} \end{pmatrix} = 0$$

<u>Summary</u>:

- Scalar multiplication and dot product can help us to define the equations of lines and Planes w.r.. Dot product can be a good tool for computing the angle and length.
- Cross product can help us to compute the volume of parallelogram and parallelepiped. Moreover, it's also a tool for defining the equation of Planes.